Exercise 2.3.6

(Language death) Thousands of the world's languages are vanishing at an alarming rate, with 90 percent of them being expected to disappear by the end of this century. Abrams and Strogatz (2003) proposed the following model of language competition, and compared it to historical data on the decline of Welsh, Scottish Gaelic, Quechua (the most common surviving indigenous language in the Americas), and other endangered languages.

Let X and Y denote two languages competing for speakers in a given society. The proportion of the population speaking X evolves according to

$$\dot{x} = (1-x)P_{YX} - xP_{XY}$$

where $0 \leq x \leq 1$ is the current fraction of the population speaking X, $1-x$ is the complementary fraction speaking Y, and P_{YX} is the rate at which individuals switch from Y to X. This deliberately idealized model assumes that the population is well mixed (meaning that it lacks all spatial and social structure) and that all speakers are monolingual.

Next, the model posits that the attractiveness of a language increases with both its number of speakers and its perceived status, as quantified by a parameter $0 \leq s \leq 1$ that reflects the social or economic opportunities afforded to its speakers. Specifically, assume that $P_{YX} = sx^a$ and, by symmetry, $P_{XY} = (1-s)(1-x)^a$, where the exponent $a > 1$ is an adjustable parameter. Then the model becomes

$$\dot{x} = s(1-x)x^a - (1-s)x(1-x)^a.$$

a) Show that this equation for $\dot{x}$ has three fixed points.

b) Show that for all $a > 1$, the fixed points at $x = 0$ and $x = 1$ are both stable.

c) Show that the third fixed point, $0 < x^* < 1$, is unstable.

This model therefore predicts that two languages cannot coexist stably—one will eventually drive the other to extinction. For a review of generalizations of the model that allow for bilingualism, social structure, etc., see Castellano et al. (2009).

Solution

Part (a)

The fixed points occur where $\dot{x} = 0$.

$$s(1-x^*)x^{*a} - (1-s)x^*(1-x^*)^a = 0$$

$$x^*(1-x^*)\left[sx^{*a-1} - (1-s)(1-x^*)^{a-1}\right] = 0$$

$$x^* = 0 \quad \text{or} \quad 1 - x^* = 0 \quad \text{or} \quad sx^{*a-1} - (1-s)(1-x^*)^{a-1} = 0$$

$$x^* = 0 \quad \text{or} \quad x^* = 1 \quad \text{or} \quad sx^{*a-1} = (1-s)(1-x^*)^{a-1}$$

Solve this third equation for x^*.

$$\frac{s}{1-s} = \frac{(1-x^*)^{a-1}}{x^{*a-1}}$$

$$\frac{s}{1-s} = \left(\frac{1-x^*}{x^*}\right)^{a-1}$$

$$\left(\frac{s}{1-s}\right)^{1/(a-1)} = \frac{1-x^*}{x^*} = \frac{1}{x^*} - 1$$

$$\frac{1}{x^*} = \left(\frac{s}{1-s}\right)^{1/(a-1)} + 1$$

$$x^* = \frac{1}{\left(\frac{s}{1-s}\right)^{1/(a-1)} + 1}$$

Part (b)

Use linear stability analysis (from the next section) to determine the stability of $x^* = 0$ and $x^* = 1$. If

$$f(x) = s(1-x)x^a - (1-s)x(1-x)^a,$$

then

$$\begin{aligned} f'(x) &= s[(-1)x^a + a(1-x)x^{a-1}] - (1-s)[(1-x)^a - ax(1-x)^{a-1}] \\ &= sx^{a-1}(-x + a - ax) + (1-s)(1-x)^{a-1}(-1 + x + ax) \\ &= sx^{a-1}[a - x(1+a)] + (1-s)(1-x)^{a-1}[-1 + x(1+a)]. \end{aligned}$$

As a result,

$$\begin{aligned} f'(0) &= -(1-s) < 0 \\ f'(1) &= -s < 0. \end{aligned}$$

Because the slope of the $\dot{x}$ versus x curve is negative at $x^* = 0$ and $x^* = 1$, each of these points is stable. This is true for any $a > 1$.

Part (c)

Let x^* be the third fixed point. Then

$$\begin{aligned} f'(x^*) &= sx^{*a-1}[a - x^*(1+a)] + (1-s)(1-x^*)^{a-1}[-1 + x^*(1+a)] \\ &= (1-s)(1-x^*)^{a-1}[a - x^*(1+a)] + (1-s)(1-x^*)^{a-1}[-1 + x^*(1+a)] \\ &= (1-s)(1-x^*)^{a-1}[a - x^*(1+a) - 1 + x^*(1+a)] \\ &= (1-s)(1-x^*)^{a-1}(a-1) > 0. \end{aligned}$$

Because the slope of the $\dot{x}$ versus x curve is positive at x^*, this point is unstable.